Recursive Statistical Analysis of the GARCH (1,1) Process for Stock Prices

F. Chan and M. McAleer

Department of Economics, University of Western Australia, (fchan@ecel.uwa.edu.au)

Abstract: Bollerslev's Generalised Autoregressive Conditional Heteroscedasticity (GARCH) Model is one of the most widely used empirical models in the literature on financial volatility. However, some of its important structural and statistical properties, such as stationarity, ergodicity, consistency and asymptotic normality, which can be shown to hold under certain moment conditions, were not established until very recently. Despite their importance in assessing the quality of any given empirical model for data analysis and forecasting, these conditions have routinely been ignored in the empirical finance literature. This paper provides empirical evidence to demonstrate the need to examine the regularity conditions. Conditions for the existence of the second and fourth moments for GARCH (1,1) are examined using two important stock indexes, namely Nikkei 225 Index and Hang Seng Index. The effects of extreme observations and outliers on the regularity conditions are also examined using rolling samples. These effects are also analysed using a simple trimming method which is designed to reduce the effects of outliers and extreme observations. The results show that the quasi-maximum likelihood estimators are sensitive to the presence of outliers, so the existence of moments is problematic over various sub-samples of the data. Consequently, two conditions which are typically assumed in the literature, namely stationarity and asymptotic normality, require further consideration in empirical analysis of financial volatility.

Keywords: GARCH; Recursive estimation; Regularity conditions; Extreme observations; Outliers

1 INTRODUCTION

Bollerslev's [1986] Generalised Autoregressive Conditional Heteroscedasticity (GARCH) model is one of the most widely used empirical models in the literature on financial volatility. Regardless of its popularity, the structural and statistical properties of this model were fully developed only recently. Moreover, the full statistical properties of the (Quasi) Maximum Likelihood Estimator ((Q)MLE) of GARCH models were also previously unknown. The lack of statistical and structural properties of the models and their estimators had made it difficult to determine the empirical adequacy of the models and the validity of any associated inferences.

Regularity conditions for stationarity, egodicity, consistency and asymptotic normality of GARCH(p, q) models have been fully developed recently by Ling and McAleer [2001a,b] (see also Bougerol and Picard [1992] and Nelson[1990]). They showed that these conditions are closely related to the existence of moments of the errors, wherein the existence of moments provides sufficient conditions for stationarity, ergodicity, consistency and asymptotic normality. Furthermore, Ling and Li [1997] and Ling and McAleer [2001a] derived the necessary and sufficient conditions for the existence of the $2m$th moments for GARCH(p, q) models. The significance of their contributions is not restricted to the development of these regularity conditions, as their results provide very simple conditions for stationarity, ergodicity, consistency and asymptotic normality for GARCH(1,1) models, which can easily be checked in practice. Thus, the validity of inferences can also be verified.

The aim of this paper is to examine the effects of extreme observations and outliers on the moment conditions. The link between the data and the empirical moment conditions is based on the fact that the moment conditions are functions of the parameters of the models. As the estimates of these parameters are influenced by the presence of outliers and extreme observations, it is important to analyse the impact of their presence on the estimates and the empirical moment conditions.

The plan of the paper is as follows. Section 2 provides a brief summary regarding the statistical and structural properties of the GARCH$(1, 1)$ model. Section 3 describes the methodology used in the paper. The empirical results are given in Sections 4 and 5. Section 6 contains some concluding remarks.

2 AR(1)-GARCH(1,1) MODEL

Consider the AR(1)-GARCH(1,1) model:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \qquad |\phi_1| < 1 \qquad (1)$$

where

$$\varepsilon_t = \eta_t \sqrt{h_t}, \qquad (2)$$

$$h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}, \qquad \omega > 0, \alpha \geq 0, \beta \geq 0. \qquad (3)$$

In (1) and (3), the parameters are typically estimated by maximum likelihood method to obtain (Quasi) Maximum Likelihood Estimators ((Q)MLE). The conditional log-likelihood function is given as follows:

$$\sum_t l_t = -\frac{1}{2}\sum_t (log h_t + \frac{\varepsilon_t^2}{h_t}). \qquad (4)$$

Ling and Li [1997] showed that the local QMLE for GARCH(p,q) is consistent and asymptotic normal if $E(\varepsilon_t^4) < \infty$, and the model is stationary and ergodic if $E(\varepsilon^2) < \infty$. Using results from Ling and Li [1997] and Ling and McAleer [2001a,b] (see also Bollerslev [1986], Nelson [1990] and He and Teräsvirta [1999]), the necessary and sufficient conditions for the existence of the second and fourth moments of ε_t are $\alpha + \beta < 1$ and $(\alpha+\beta)^2 + 2\alpha^2 < 1$, respectively.

3 METHODOLOGY

3.1 Data

In this paper, AR(1)-GARCH(1,1) models are estimated using two different stock indexes, namely Nikkei 225 Index and Hang Seng Index. The data were obtained through the DataStream database service. The sample is from 1/1/1986 to 11/4/2000, with a total of 3725 observations for each index, and the stock returns are calculated as

$$R_t = \frac{Y_t - Y_{t-1}}{Y_{t-1}}.$$

3.2 Procedures

The AR(1)-GARCH(1,1) specification defined in (1) - (3) is used to model the returns of each index. These models are estimated in section 4 for a rolling window of size 750 for each index. The impact of each observation on the estimates and on the second and fourth moment conditions can be examined by the dynamic path of the estimates.

A trimming method is applied to each rolling window to mitigate the effects of extreme observations and outliers. The GARCH(1,1) models are re-estimated using the adjusted data, and the effects of extreme observations and outliers are investigated by comparing the dynamic paths of the estimates using the adjusted and unadjusted data.

3.3 Trimming Method

A trimming method is applied to each rolling window. The algorithm can be summarised as follows:

1. Calculate the standard deviation of the sample.
2. If an observation is 4 times larger than the standard deviation, it is reduced to 4 times the standard deviation.
3. If an observation is between 3 and 4 times the standard deviation, it is reduced to 3 times the standard deviation.
4. If an observation is between 2.5 and 3 times the standard deviation, it is reduced to 2.5 times the standard deviation.
5. Repeat steps 1 to 4 above for each observation in the sample.

4 EMPIRICAL RESULTS

The empirical results for the GARCH(1,1) models are as follows:

4.1 Nikkei 225

As shown in figure 1 in Appendix A, the movements in $\hat{\alpha}$ for Nikkei 225 appear to be dramatic. The market crash of October 1987 appears to have had a great impact on both $\hat{\alpha}$ and $\hat{\beta}$ for Nikkei, with $\hat{\alpha}$ being as high as 0.564 when October 1987 is included in the rolling sample. The effect of this outlier is even more obvious when the adjusted data are examined. All $\hat{\alpha}$ estimates with adjusted data are below 0.2, but the minimum values of $\hat{\alpha}$ are very similar for adjusted and unadjusted data.

Movements in $\hat{\beta}$ suffer similar impacts from the same outlying observation, with $\hat{\beta}$ being as low as 0.284 when October 1987 is included in the rolling sample. However, the minimum value of $\hat{\beta}$ increases to 0.769 when the outlier is reduced.

Outliers seem to have a positive impact on short run persistence and a negative impact on long run persistence. This is supported by the changes in $\hat{\alpha}$ and $\hat{\beta}$ when the outliers are removed from the rolling samples. Moreover, $\hat{\alpha}$ with the adjusted data remains low while $\hat{\beta}$ remains high. This result is consistent with other findings in the literature (see, for example, Verhoven and McAleer [1999])

The number of rolling samples failing to satisfy the second moment condition is frightening. A total of 632 rolling samples do not satisfy the second moment condition, which is reduced dramatically to 257 after the data are adjusted (see Table 1). The effects of outliers on the second moment condition can be seen by comparing figures 5 and 6. Differences between the adjusted and unadjusted data for the first 500 rolling samples represent the effects of outliers on the second moment condition. As shown in figure 5, the first 500 rolling samples are extremely volatile, with most not satisfying the second moment condition. This is no longer true after the data are adjusted, as shown in figure 6.

The situation with the fourth moment condition is equally dramatic. A total of 1073 rolling samples fail to satisfy the fourth moment condition, which is reduced to 588 after the data are adjusted (see Table 1).

Rolling samples 600 to 1000 seem to be problematic as they fail to satisfy the second and fourth moment conditions for both the adjusted and unadjusted data. This outcome suggests two possibilities: (i) GARCH(1,1) might not be an appropriate model for these observations; (ii) the effects of outliers and extreme observations are still high, suggesting that the trimming algorithm is not successful for these samples. Such an outcome can arise particularly when a substantial number of outliers and extreme observations is present in a rolling sample. An excessive number of extreme observations will lead to a high standard deviation, thereby decreasing the number of observations being adjusted by the algorithm. Combining this possibility with the magnitude of the adjustment can explain the sudden changes in the estimated $\hat{\alpha}$ and $\hat{\beta}$.

4.2 Hang Seng

As compared with the previous index, the movements in $\hat{\alpha}$ and $\hat{\beta}$ are very different for Hang Seng. The most distinct difference is the large negative shock on 4 June 1989, the Tianenman Square incident, which caused a great upset in the Hong Kong market. This observation led to the second largest decrease in $\hat{\alpha}$ when it was removed from the rolling sample, as shown in figure 9. Moreover, this outlier seems to have had a greater impact on $\hat{\beta}$ estimates than October 1987, which is supported by the relatively large decrease in $\hat{\beta}$ in figure 11.

As shown in figures 10 and 12, the adjusted data seem to have removed the impact of October 1987, though the effect of 4 June 1989 largely remains. This outcome explains the sudden increase in $\hat{\alpha}$ in the early rolling samples in figure 10. There seems to be little difference in the $\hat{\beta}$ estimates between the unadjusted and adjusted data.

Although the Hang Seng returns seem more dramatic than the Nikkei returns, it is the only index which statisfies the second moment condition for all rolling samples when the data are adjusted. A total of 640 rolling samples from the unadjusted data fail to satisfy the second moment condition (see Table 1). For the fourth moment condition, a total of 1178 rolling samples fail to satisfy the condition for unadjusted data, with 304 failures being recorded for adjusted data (see Table 1).

As shown in figures 14 and 16, reducing the magnitude of outliers tends to reduce the number of sudden changes, and also narrows the range of variation in both conditions. This is especially transparent in figure 16, where the maximum value of the fourth moment condition is 1.49 for unadjusted data, but only 1.03 for adjusted data.

5 REGULARITY CONDITIONS

Information about the second and fourth moment condition is summarised in Table 1, where the figures represent the number of rolling samples not satisfying the regularity conditions for both the adjusted and unadjusted data.

These figures highlight two important points. First, examination of the regularity conditions is essential in estimating any GARCH model. Over 25% of the rolling samples fail to satisfy the second moment condition with GARCH for Nikkei and Hang Seng, meaning that GARCH(1, 1) may not be stationary for these rolling samples, and the estimates may not be consistent, especially for unadjusted data. Worse still, more than one third of the rolling samples fail to satisfy the fourth moment condition, which means the QMLE may not be asymptotic normal for these rolling samples. Therefore, the validity of inferences is doubtful in more than one third of the cases for rolling samples for both Nikkei and Hang Seng!

Outliers and extreme observations play an important role in satisfying the regularity conditions. Reducing the effects of outliers seems to have a positive effect on the number of rolling samples satisfying the moment conditions. It is not necessarily the case that the rolling samples failing to satisfy the moment condition for adjusted data are a subset of those failing to satisfy the condition for unadjusted data. Reducing the effects of outliers through data adjustment may affect the adequacy of the model for a particular sample, which may explain why some samples satisfy the moment conditions before trimming, but not thereafter. Thus, outliers should be handled with care.

Valid inferences play a crucial role in testing economic and financial theories using empirical evidence. Much research in modelling financial volatility has ignored the importance of valid inferences and their associated assumptions in examining the statistical significance of estimates. The aim of this paper has been to demonstrate that the assumptions essential for valid inference are often violated, either through the presence of outliers or because of other problems such as model specification, which have not yet been adequately investigated.

	Second Moment Unadjusted/Adjusted	Fourth Moment Unadjusted/Adjusted
Nikkei	632/257	1073/558
Hang Seng	640/0	1178/304

Table 1: Number of rolling windows that fail to satisfy the second and fourth moment conditions for GARCH(1, 1)

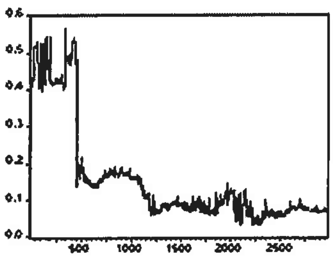

Figure 1: Rolling $\hat{\alpha}$ for unadjusted Nikkei

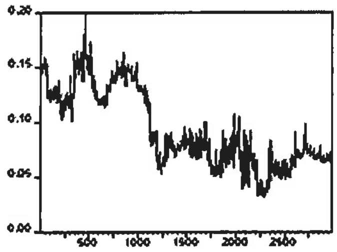

Figure 2: Rolling $\hat{\alpha}$ for adjusted Nikkei

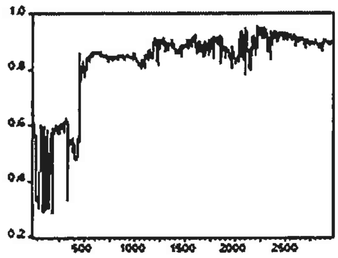

Figure 3: Rolling $\hat{\beta}$ for unadjusted Nikkei

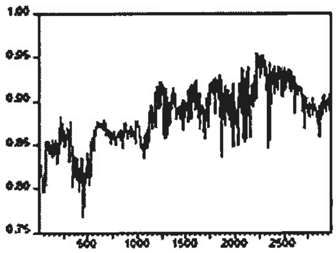

Figure 4: Rolling $\hat{\beta}$ for adjusted Nikkei

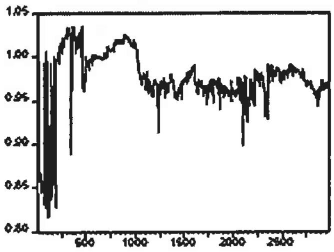

Figure 5: Rolling Second Moment for unadjusted Nikkei

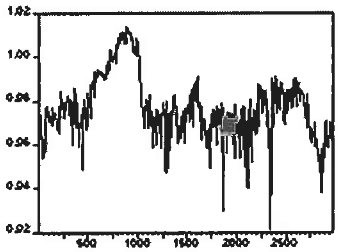

Figure 6: Rolling Second Moment for adjusted Nikkei

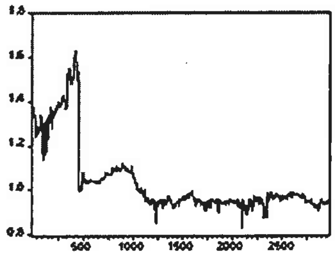

Figure 7: Rolling Fourth Moment for unadjusted Nikkei

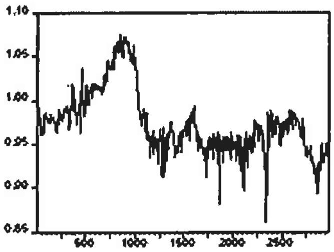

Figure 8: Rolling Fourth Moment for adjusted Nikkei

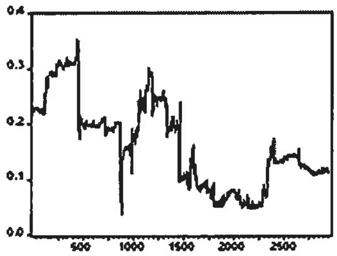

Figure 9: Rolling $\hat{\alpha}$ for unadjusted Hang Seng

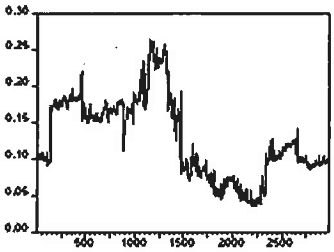

Figure 10: Rolling $\hat{\alpha}$ for adjusted Hang Seng

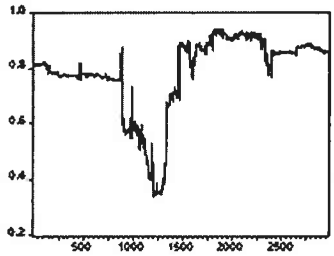

Figure 11: Rolling $\hat{\beta}$ for unadjusted Hang Seng

Figure 12: Rolling $\hat{\beta}$ for adjusted Hang Seng

Figure 13: Rolling Second Moment for unadjusted Hang Seng

Figure 14: Rolling Second Moment for adjusted Hang Seng

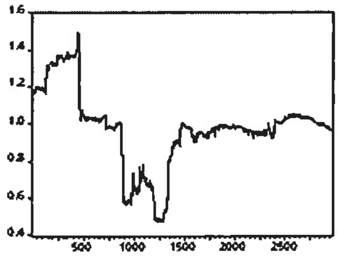

Figure 15: Rolling Fourth Moment for unadjusted Hang Seng

Figure 16: Rolling Fourth Moment for adjusted Hang Seng

6 CONCLUSION

The objectives of this paper were as to:

1. examine the effects of outliers and extreme observations on the QMLE for the GARCH(1,1) model;

2. evaluate the effects of outliers and extreme observations on the regularity conditions relating to the existence of the second and fourth moments;

3. illustrate the importance of evaluating the regularity conditions empirically to obtain valid inferences.

As shown in this paper, statistical properties such as consistency and asymptotic normality may not hold in practice, and QMLE can be highly sensitive to extreme observations and outliers. This paper illustrated the importance of examining the regularity conditions in order to ensure stationarity, consistency and valid inferences. It also provided motivation for developing statistical results regarding other popular GARCH-type models, such as Glosten, Jagannathan and Runkle's [1992] GJR-GARCH and Nelson's [1991] Exponential GARCH (EGARCH).

Further research would usefully concentrate on establishing the minimum conditions necessary for stationarity, ergodicity, consistency and asymptotic normality for other GARCH-type models such as GJR-GARCH and EGARCH. As demonstrated empirically, the number of rolling samples failing to satisfy the moment conditions increases with the order of the moment, for both unadjusted and adjusted data. This result implies that higher-order moment conditions impose more complicated restrictions on empirical models. For obvious reasons, lower moment conditions are to be preferred as they are easier to check and satisfy in practice.

7 ACKNOWLEDGEMENTS

The first author would like to acknowledge an Australian Postgraduate Award and an Individual Research Grant, Faculties of Economics and Commerce, Education and Law, University of Western Australia, for financial support. The second author wishes to acknowledge the financial support of the Australian Research Council.

REFERENCES

Bollerslev, T. Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31:307–327, 1986.

Bougerol, P. and N.M. Picard. Stationarity of GARCH Processes and of Some Non-negative Time Series. *Journal of Econometrics*, 52:115–127, 1992.

Glosten, L., R. Jagannathan, and D. Runkle. On the Relation Between the Expected Value and Volatility of Nominal Excess Return on Stocks. *Journal of Finance*, 46:1779–1801, 1992.

He, C. and T. Teräsvirta. Properties of Moments of a Family of GARCH Processes. *Journal of Econometrics*, 92:173–192, 1999.

Ling, S. and W.K. Li. On Fractional Integrated Autoregressive Moving Average Time Series Models with Conditional Heteroscedasticity. *Journal of the American Statistical Association*, 92:1184–1194, 1997.

Ling, S. and M. McAleer. Necessary and Sufficient Moment Conditions for the GARCH(r,s) and Asymmetric Power GARCH(r,s) Models. To appear in *Econometric Theory*, 2001a.

Ling, S. and M. McAleer. Stationarity and the Existence of Moments of a Family of GARCH Processes. To appear in *Journal of Econometrics*, 2001b.

Nelson, D.B. Stationarity and Persistence in the GARCH(1,1) Model. *Econometric Theory*, 6:318–334, 1990.

Nelson, D.B. Conditional Heteroscedasticity in Asset Returns: A New Approach. *Econometrica*, 59:347–370, 1991.

Verhoven, P. and M. McAleer. Modelling Outliers and Extreme Observations for ARMA-GARCH Processes. Submitted.